

Yang-Mills by dimensionally reducing Chern-Simons

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ABSTRACT

We derive the usual first-order form of the Yang-Mills action in arbitrary dimensions by dimensional reduction from a Chern-Simons-like action. The antisymmetric tensor auxiliary field of the first-order action appears as a gauge field for the extra dimensions. The higher-dimensional geometry was introduced in our previous paper by adding dimensions “dual” to spin, as suggested by the superstring’s affine Lie algebra.

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Chern-Simons analogs

We consider a class of theories related to Chern-Simons, but defined in more than three dimensions, and dependent on the geometry. The theories are defined as usual in terms of Yang-Mills covariant derivatives $\nabla_A = d_A + iA_A$, but the free derivatives d_A are nonabelian. Dividing up the derivatives as $A = (\alpha, i)$,

$$[d_A, d_B] = T_{AB}{}^C d_C : \quad [d_\alpha, d_\beta] = T_{\alpha\beta}{}^i d_i, \quad [d_\alpha, d_i] = [d_i, d_j] = 0$$

The field strengths are

$$F_{AB} = d_{[A} A_{B]} - T_{AB}{}^C A_C + iA_{[A} A_{B]}$$

Defining as usual the Chern-Simons form (but taking into account the torsion)

$$X_{ABC} = \frac{1}{2} A_{[A} d_B A_{C]} - \frac{1}{4} A_{[A} T_{BC]}{}^D A_D + \frac{1}{3} i A_{[A} A_B A_{C]}$$

the Lagrangian takes the form (appearing as its trace in the action)

$$L = \frac{1}{2} a^{\alpha\beta i} X_{\alpha\beta i}$$

for some constant tensor a invariant under the desired symmetries.

All such actions have an important difference from standard Chern-Simons terms: The torsion introduces a contribution

$$-\frac{1}{2} \eta^{ij} A_i A_j, \quad \eta^{ij} \equiv \frac{1}{4} a^{\alpha\beta(i} T_{\alpha\beta}{}^{j)}$$

We assume η is invertible, which allows A_i to be removed from the action as an auxiliary field by its field equation. The action thus resembles the first-order form of a standard Yang-Mills action:

$$L = (-\frac{1}{2} \eta^{ij} A_i A_j + A_i \hat{F}^i) - \frac{1}{2} a^{\alpha\beta i} A_\alpha d_i A_\beta, \quad \hat{F}^i \equiv \frac{1}{2} a^{\alpha\beta i} (d_{[\alpha} A_{\beta]} + i A_{[\alpha} A_{\beta]})$$

But \hat{F} is not covariant; in fact the auxiliary field equation is $\eta^{ij} A_j = \hat{F}^i$.

There are several nontrivial examples of such actions in the literature. One is the minimal supersymmetrization of the usual 3D Chern-Simons form [1]: In that case d_α are the usual 2 supersymmetry derivatives and d_i the 3 translations, while T and a are both γ matrices. The auxiliary field equation is just the usual conventional constraint determining the vector gauge field in terms of the spinor. A similar expression yields the action for 4D N=1 super Yang-Mills (with α a 4-spinor index), if the representation-preserving constraints are imposed by hand [2]. Another example is the action for maximally supersymmetric Yang-Mills in 4D N=3 harmonic superspace [3]: In that case the 2 d_α and 1 d_i are the 3 lowering operators of $SU(3)/U(1)^2$. Unlike the previous cases, not all the derivatives of the space appear in that Lagrangian.

Dimensional reduction

In previous papers [4] we derived the affine Lie algebra of the superstring, containing the generalization of the superparticle's spinor and vector derivatives, but also a spinor “dual” to that spinor derivative. The spinor field strength of super Yang-Mills appeared as a “gauge field” to the new spinor derivative, in the same way as the usual spinor and vector gauge superfields did for the other two derivatives.

In a recent paper [5] we considered the natural extension of this affine Lie algebra to include a derivative for which the antisymmetric tensor field strength is the gauge field. It appeared as a necessary consequence of including spin operators in the algebra, as their dual. (“Prepotentials” appeared as gauge fields for the spin.)

We then applied this algebra directly to super Yang-Mills and supergravity, without direct reference to strings. Superspace was derived by a combination of isotropy constraints, imposed in terms of gauge covariant derivatives (for the spin derivatives), and dimensional reduction, imposed in terms of dual symmetry generators. The former eliminated also the corresponding gauge fields, in covariant gauges, while the latter kept the corresponding gauge fields, but as field strengths (as in conventional forms of dimensional reduction).

In this paper we consider only bosonic Yang-Mills, so we limit our algebra of derivatives to only the usual translations and the derivatives dual to spin. The algebra is a special case of that considered in the previous section:

$$[d_a, d_b] = d_{ab}, \quad [d_{ab}, d_c] = [d_{ab}, d_{cd}] = 0$$

with $\alpha = a$, $i = ab$. We now have

$$T_{a,b}{}^{cd} = \frac{1}{2}\delta_{[a}^c\delta_{b]}^d, \quad a^{a,b,cd} = \eta^{a[c}\eta^{d]b}, \quad \eta^{ab,cd} = \frac{1}{2}\eta^{a[c}\eta^{d]b}$$

$$\hat{F}^{ab} = d^{[a}A^{b]} + iA^{[a}A^{b]}$$

In our previous paper we did not consider an action for the theory before dimensional reduction, although this is generally how this reduction is applied, especially in supersymmetric theories. We now see that the Chern-Simons-like action above is the one suited for this purpose: (1) It directly identifies the gauge field A_{ab} for the dual spin coordinates as the usual field strength \hat{F} , which is covariant upon reduction because the noncovariant d_{ab} term in its transformation law dies. (2) It gives the usual first-order Yang-Mills action upon reduction, since the extra $A_a d^{ab} A_b$ term drops out.

In D=4, the extra dimensions (and thus the extra field) can be restricted to be self-dual.

Conclusions

We have shown a natural way of introducing first-order formalisms, where the auxiliary field is originally nontrivial, appearing as a gauge field. This field (and the corresponding extra dimensions) is also suggested by an algebraic analysis of the superstring. This approach might have advantages similar to those of the manifestly T-dual formulation of the actions for low energy states of strings, which is treated as dimensional reduction from twice the coordinates [6].

Generalization to higher spins (e.g., gravity) and supersymmetry should be considered. For example, the usual auxiliary fields of 4D N=1 and 2 super Yang-Mills (dimension-2 scalars in the adjoint of U(1) and SU(2) R-symmetry, respectively) would appear as the gauge fields for extra dimensions dual to R-symmetry (instead of spin), since they appear in generalized d'Alembertians as the R-symmetry analog of the relativistic Pauli term. Another possible application would be to find a first-order formalism for string field theory, with on-shell field strengths built into the formalism. A first-quantized formulation of the approach would be a first step.

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